Definition (normed space)

Let X be a vector space over a field $\mathbb{K}$ and $\|\cdot\|: X \to \mathbb{R}$ be a function. $(X, \|\cdot\|)$ is called a normed space if

(i) $\|x\| \geq 0$ for all $x \in X$.

(ii) $\|x\| = 0$ if and only if $x = 0$.

(iii) $\|\alpha x\| = |\alpha| \|x\|$ for $x \in X$ and $\alpha \in \mathbb{K}$.

(iv) $\|x - y\| \leq \|x - z\| + \|z - y\|$ for $x, y, z \in X$.

Example

Write a sequence of numbers as a function $x : \{1, 2, \dots\} \to \mathbb{K}$

(a) $\ell_\infty := \{ x(i) : x(i) \in \mathbb{K} , \sup_i |x(i)| < \infty \}$ and

$$\|x\|_\infty := \sup_i |x(i)|$$

(i) For any $x \in \ell_\infty$, $\|x\|_\infty = \sup_i |x(i)| \geq 0$

(ii) $\|x\|_\infty = 0 \iff \sup_i |x(i)| = 0$

$\iff |x(i)| = 0$ for all $i = 1, 2, \dots$

$\iff x(i) = 0$ for all $i = 1, 2, \dots$

$\iff x = (0, 0, \dots)$

(iii) For any $x \in l_\infty$ and $\alpha \in \mathbb{K}$,

$$\begin{aligned}\|\alpha x\|_\infty &= \sup_i |\alpha x(i)| \\ &= \sup_i |\alpha|\,|x(i)| \\ &= |\alpha| \sup_i |x(i)| \\ &= |\alpha|\,\|x\|_\infty\end{aligned}$$

(iv) For any $x, y, z \in l_\infty$,

$$\begin{aligned}\|x-y\|_\infty &= \sup_i |x(i) - y(i)| \\ &\leq \sup_i \left(|x(i) - z(i)| + |z(i) - y(i)| \right) \\ &\leq \sup_i |x(i) - z(i)| + \sup_i |z(i) - y(i)| \\ &= \|x-z\|_\infty + \|z-y\|_\infty\end{aligned}$$

[$\sup_i (a_i + b_i) \leq \sup_i a_i + \sup_i b_i$
MATH 2050, definition:
supremum is the least upper bound]

Hence, $(l_\infty, \|\cdot\|_\infty)$ is a normed space.

(b) For $1 \le p < \infty$, put $\ell_p := \left\{ (x(i)) : x(i) \in \mathbb{K},\ \sum_{i=1}^{\infty} |x(i)|^p < \infty \right\}$ and $\|x\|_p := \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}}$.

(i) For any $x \in \ell_p$, $\|x\|_p := \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}} \ge 0$

(ii) $\|x\|_p = 0 \iff \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}} = 0$

$$\iff \sum_{i=1}^{\infty} |x(i)|^p = 0$$

$$\iff |x(i)| = 0 \quad \text{for all } i = 1, 2, \ldots$$

$$\iff x(i) = 0 \quad \text{for all } i = 1, 2, \ldots$$

$$\iff x = (0, 0, \ldots).$$

(iii) $\|\alpha x\|_p = \left(\sum_{i=1}^{\infty} |\alpha x(i)|^p \right)^{\frac{1}{p}}$

$$= \left(\sum_{i=1}^{\infty} |\alpha|^p |x(i)|^p \right)^{\frac{1}{p}}$$

$$= \left(|\alpha|^p \sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}}$$

$$= \left(|\alpha|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}}$$

$$= |\alpha| \, \|x\|_p$$

(iv) $\boxed{\begin{array}{l}\text{Minkowski Inequality:} \\ \text{For any } x, y \in \ell_p,\ \|x+y\|_p \leq \|x\|_p + \|y\|_p .\end{array}}$

For any $x, y, z \in \ell_p$, write $x' = x - z$ and $y' = z - y$.

Then $\|x-y\|_p = \|x'+y'\|_p$

$\leq \|x'\|_p + \|y'\|_p$ by Minkowski

$= \|x-z\|_p + \|z-y\|_p$.

Therefore, it suffices to prove Minkowski inequality.

Proof of Minkowski Inequality:

When $p=1$, it follows immediately from triangle inequality on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

When $1 < p < \infty$, set $q = \frac{p}{p-1}$ (or $\frac{1}{p} + \frac{1}{q} = 1$). Recall

$$\boxed{\begin{array}{l}\text{Hölder Inequality} \\ \displaystyle\sum_{i=1}^{\infty} |x(i)||y(i)| \leq \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y(i)|^q\right)^{\frac{1}{q}}\end{array}}$$

Pick any $x, y \in l_p$

Put $x'(i) = x(i)$ and $y'(i) = |x(i)+y(i)|^{p-1}$. Then

$$\sum_{i=1}^{\infty} |x(i)|\,|x(i)+y(i)|^{p-1} \leq \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$= \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{\frac{1}{q}}$$

Similarly,

$$\sum_{i=1}^{\infty} |y(i)|\,|x(i)+y(i)|^{p-1} \leq \left(\sum_{i=1}^{\infty} |y(i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{\frac{1}{q}}$$

Combine these two inequalities, we get

$$\sum_{i=1}^{\infty} |x(i)+y(i)|^p = \sum_{i=1}^{\infty} |x(i)+y(i)|\,|x(i)+y(i)|^{p-1}$$

$$\leq \sum_{i=1}^{\infty} (|x(i)|+|y(i)|)\,|x(i)+y(i)|^{p-1}$$

$$= \sum_{i=1}^{\infty} |x(i)|\,|x(i)+y(i)|^{p-1} + \sum_{i=1}^{\infty} |y(i)|\,|x(i)+y(i)|^{p-1}$$

$$\leq \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{\frac{1}{q}}$$

$$+ \left(\sum_{i=1}^{\infty} |y(i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{\frac{1}{q}}$$

$$= (\|x\|_p + \|y\|_p) \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{\frac{1}{q}}$$

$$\Rightarrow \|x+y\|_p = \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{\infty} |x(i)+y(i)|^p\right)^{1-\frac{1}{q}}$$

$$\leq \|x\|_p + \|y\|_p .$$

$\square$

Proof of Hölder's Inequality:

Recall

Young's Inequality If $\frac{1}{p}+\frac{1}{q}=1$, $a,b \geq 0$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

(You can check it by basic calculus)

We first assume $\|x\|_p = \|y\|_q = 1$

$$\sum_{i=1}^{\infty} |x(i)||y(i)| \leq \sum_{i=1}^{\infty}\left(\frac{|x(i)|^p}{p} + \frac{|y(i)|^q}{q}\right)$$

$$= \frac{1}{p}\sum_{i=1}^{\infty} |x(i)|^p + \frac{1}{q}\sum |y(i)|^q$$

$$= \frac{1}{p}\|x\|_p^p + \frac{1}{q}\|y\|_q^q = 1 .$$

In general case, put $x' = \frac{x}{\|x\|_p}$, $y' = \frac{y}{\|y\|_q}$.

Then $\sum_{i=1}^{\infty} \frac{|x(i)|}{\|x\|_p} \frac{|y(i)|}{\|y\|_q} \leq 1$, i.e., $\sum_{i=1}^{\infty} |x(i)||y(i)| \leq \|x\|_p \|y\|_q$.

$\square$